NONSTEADY MOTION OF A TRANSVERSE-SHEAR CRACK ACROSS THE INTERFACE BETWEEN ELASTIC MEDIA

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This article examines the motion of a crack along the line joining two different elastic half-planes under the influence of variable shear stresses. Analogous to the case of a homogeneous medium [1-3], the law of motion of the edge is assumed to be known. Among the features of the physical situation being examined are the nonsymmetrical character of the solution with a symmetrical load distribution and the dependence of the number of Rayleigh wave which can be generated (two, one, none) on the ratios of the elastic parameters. The problem decomposes in the image space into a scalar problem of conjugating two functions reflecting the connection between the displacement discontinuity on the crack and the shear stress on the crack extension. The formula must then be inverted to represent the normal stress. The solution is constructed by the method of factorization, which was used in [2, 3] for a problem with a movable separation point for the boundary conditions. The properties of the Rayleigh boundary function for contacting elastic bodies are also studied. It is shown that the Hölder continuity condition for the input functions is sufficient to determine the asymptotes at the edge of the crack, analogous to the case of steady crack movement [4]. With transformations of the convolutions, we used the methods of contour integration and applied the residue theorem. This made it possible to somewhat simplify the results [2]. The subject of crack starting is addressed in an examination of special types of loading. The solution of a similarity problem was given in [5].

1. Formulation and General Solution of the Problem. A semi-infinite notch (transverse shear crack) is located on the rectilinear boundary between two elastic planes. Beginning at the moment of time t = 0, the edge of the notch moves in accordance with the law $x_1 = \ell(t)$, $\dot{\ell} \geq 0$ [$\dot{\ell} = d\ell/dt$, $\ell(0) = L$]. The contacting edges of the notch offer no resistance to shear; conditions of complete contact are satisfied at $x_1 \geq \ell(t)$, $x_2 = 0$ (x_1 and x_2 constitute a Cartesian coordinate system).

We will examine the external dynamic stress and displacement fields, which exist (and are known) as the solution of the corresponding plane problem without a crack. The additional stress field $\sigma_{km}(x_1, x_2, t)$ and displacement field $u_k(x_1, x_2, t)$ which develop as a result of diffraction must satisfy the dynamic equations of the linear theory of elasticity ($x_2 \neq 0$) and the following boundary conditions ($x_2 = 0$, $x \equiv x_1$):

$$\begin{gathered}\sigma_{12} = \tau_-(x, t),\ [\sigma_{12}] = [\sigma_{22}] = [u_2] = 0,\ x < l, \\ [\sigma_{12}] = [\sigma_{22}] = [u_1] = [u_2] = 0,\ x > l, \\ 0 \leqslant W < \infty \Rightarrow |\sigma_{km}| < C_0\,[(x-l)^2 + x_2^2]^{-1/4},\ x \to l,\ x_2 \to 0 \\ (k, m = 1, 2).\end{gathered} \tag{1.1}$$

Here, the square brackets denote a discontinuity of a quantity in the transition across the interface between the elastic planes; W is the flow of energy to the edge of the notch; $C_0 > 0$ is a constant.

We adopt zero initial conditions. We also assume that continuity of the displacement u_2 at $x < \ell$, x = 0 (i.e., the condition of non-negativity of the contact pressure) can be assured by adding the static uniform stress state to the solution as a whole.

The above-formulated initial-boundary-value problem for linear hyperbolic equations can be examined in different classes of input functions up to distributions (fundamental solutions). The uniqueness of the solution for smooth functions τ_-, $\ell\cdot$ can be demonstrated

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similarly to [6, 7]. Here, the requirement of non-negativity of the energy flow W is important.

We will use capital letters to designate Fourier transformation with respect to x and Laplace transformation with respect to t with the parameters q and s. Proceeding on the basis of the relations between the images of the displacements and stresses on the boundary of the elastic half-plane [3] and taking conditions (1.1) into consideration, we exclude U(q, s) and Σ(q, s) [the images of the functions $\sigma(x, t) = \sigma_{22}(x, 0, t)$ and $u(x, t) = u_2(x, 0, t)$] and we obtain a problem of conjugation of the functions $F_-(q, s)$ and $T_+(q, s)$ – the images of the discontinuity of the displacements $f_- = [u_1]$ on the notch and the shear stresses on the notch continuation $\tau_+ = \sigma_{12}(x, 0, t)H(x - \ell)$:

$$\begin{gathered} F_- = G(T_+ + T_-), \\ qG = S/P,\ P(i\xi) = 2\mu_1[\beta_{11}(1-\beta_1)R_2 + \mu\beta_{12}(1-\beta_2)R_1], \\ S(i\xi) = (\mu-1)^2\beta_{11}\beta_{12}\beta_{21}\beta_{22} + (\mu\beta_1-\beta_2)^2 - \beta_{11}\beta_{21}(\mu-\beta_2)^2 - \\ - \beta_{12}\beta_{22}(1-\mu\beta_1)^2 - \mu(1-\beta_1)(1-\beta_2)(\beta_{11}\beta_{22}+\beta_{12}\beta_{21}), \\ R_j(i\xi) = \beta_{1j}\beta_{2j} - \beta_j^2,\ \beta_{kj} = \sqrt{1+(c_{kj}\xi)^{-2}},\ 2\beta_j = 1 + \beta_{2j}^2, \\ \xi = q/s,\ \mu = \mu_1/\mu_2 \quad (\tau_-(x, t) = 0,\ x > l). \end{gathered} \tag{1.2}$$

Here, H(x) is the Heaviside function; μ_j are shear moduli; c_{1j} and c_{2j} are the velocities of the rarefaction and shear waves; the index j = 1, 2 denotes media 1 and 2 occupying the half-planes $x_2 > 0$ and $x_2 < 0$.

After we solve problem (1.2), we determine the stress σ and the displacement u by inversion of Eqs. (1.3):

$$\begin{gathered} \Sigma = iqDF_-/S,\ U = iEF_-/S, \\ D(i\xi) = 2\mu_1[(\beta_{11}\beta_{21}-\beta_1)R_2 - (\beta_{12}\beta_{22}-\beta_2)R_1], \\ E(i\xi) = \mu(\beta_{12}(1-\beta_2)(\beta_{11}\beta_{21}-\beta_1) + \beta_{11}(1-\beta_1)(\beta_{12}\beta_{22}-\beta_2)]. \end{gathered} \tag{1.3}$$

We make the radicals $\beta_{kj} = \xi^{-1}\sqrt{\xi^2 + c_{kj}^{-2}}$ uniform by making notches $]-\infty, -ic_{kj}^{-1}] \cup [ic_{kj}^{-1}, \infty[$ in the ξ-plane and assuming $\sqrt{\xi^2 + c_{kj}^{-2}} = c_{kj}^{-1}$ at ξ = 0.

We will limit ourselves to searching for (and analyzing) the functions τ_+, f_-, σ, and u, the images of which satisfy Eqs. (1.2) and (1.3). When the transforms permit, they are equivalent to all of the conditions (1.1) except for the energy condition. The last estimate in (1.1) is taken into consideration during the construction of the solution [1-3].

Let us attempt to find a representation of the coefficient of problem (1.2) in the form

$$G = G_+G_-,\ P_\pm = 1/G_\pm \tag{1.4}$$

with the condition that the originals of the functions G_+, P_+ vanish at $x < v_1t$ and the originals of the functions G_-, P_- vanish at $x > v_2t$ ($v_2 < \ell\cdot(t) < v_1$) [2]. First we study the character and location of singular points of the functions G and G^{-1}. We introduce the notation

$$a = \min c_{1j}^{-1},\ b = \max c_{2j}^{-1},\ c_R = \min c_{Rj},\ c_R^* = \max c_{Rj}, \quad j = 1, 2,$$

where c_{Rj} are unique positive roots of the Rayleigh equations $R_j(v^{-1}) = 0$.

The following were proven in [8]: 1) with small real values of $v = (i\xi)^{-1}$, the function $S(v^{-1}) < 0$. If $S(b) \geq 0$, then there exist simple (unique in the plane ξ) zeros of this function $v = \pm s$, $0 < c_S \leq b^{-1}$ being the velocities of Strouhal boundary waves propagating along the line of contact of dissimilar elastic media; 2) if $S(b) < 0$, then the function S(iξ) has no zeros in the complex (cut) plane ξ.

The presence of real zeros for the function P(iξ) indicates the existence of Rayleigh boundary waves propagating along the line of contact of different elastic media with slip [9]. Using the argument principle [10], we can augment [8, 9] by proving the following: 3) with small real values v of the function $P(v^{-1}) > 0$ and if $P(b) \leq 0$, then there are simple (and unique) zeros $v = \pm c_P$ of this function. Here, the value of c_P lies within the interval $c_R < c_P < c_R^*$ if $c_R^* \leq b^{-1}$ and in the half-interval $c_R < c_P \leq b^{-1}$ if $c_R^* > b^{-1}$; 4) with the condition $P(b) > 0$, this function has no zeros in the plane ξ; 5) at $c_{11} = c_{12}$, it vanishes (order of 1/2) at the limit at branch points $v = \pm c_{11}$; 6) if there exists a zero c_S($S(b) \geq 0$), then there exists a zero c_P. Meanwhile, $c_P < c_S \leq b^{-1}$ (case 1). If $S(b) < 0$, i.e., if there is no Strouhal boundary wave, then there is a region in the space of allowable elastic parameters of media 1 and 2 where $P(b) \leq 0$ (case 2). Finally, in the region of elastic parameters where the inequality $P(b) > 0$ is satisfied, $S(b) < 0$, i.e., neither type of boundary wave is present (case 3).

It should be noted that if $c_R^* \leq b^{-1}$ (transverse waves of similar velocity in the medium), the zero c_P always exists. It can be absent only when $b^{-1} < c_R^*$, i.e., for the case of materials with substantially different properties – such as when one medium is rigid [9].

Let us determine the zeros, poles, and branch point $\xi = \infty$ of the function G by representing it in the form of a product

$$G = AG_0G_1,\ G_1 = G/(AG_0),$$
$$A = -\frac{1}{4\mu_1}(3-4\nu_1+\mu)(3\mu+1-4\mu\nu_2)/(1-\nu_1+\mu-\mu\nu_2) \tag{1.5}$$

(ν_j are the Poisson's ratios).

In accordance with theorem (6) above, we choose the following in cases 1-3

$$G_0 = \frac{h^2+\xi^2}{d^2+\xi^2}\frac{s^{-1}}{\sqrt{b^2+\xi^2}},\ G_0 = s^{-i}\frac{\sqrt{b^2+\xi^2}}{d^2+\xi^2},\ G_0 = \frac{s^{-1}}{\sqrt{b^2+\xi^2}},$$
$$h = c_S^{-1},\quad d = c_P^{-1}. \tag{1.6}$$

Each subsequent case in (1.6) is obtained from the preceding case by means of the transitions: $h \to b$ ($1 \to 1$), $d \to b$ ($2 \to 3$). This theorem remains valid in relation to the solution of the problem as well (direct proof). Thus, all of the discussions below will be conducted for the most general case 1, when both of the above-mentioned Rayleigh boundary waves are present. The mathematical transition $1 \to 3$ is accompanied by a reduction in the number of poles of the function G_0 or G_0^{-1}.

The function $G_1(i\xi)$ which remains after isolation of the singularities is regular in the plane with cuts $[\pm ia, \pm ib]$ along the imaginary axis $G_1 = 1 + O(\xi^{-2})$ at $\xi \to \infty$. Thus, the function G_1 can be factored in accordance with the rule [3]

$$G_1 = \varkappa_+(i\xi)\varkappa_-(i\xi),\ \varkappa_\pm = \exp\left\{\frac{1}{\pi}\int_a^b \frac{\varphi(\alpha)\,d\alpha}{\alpha \mp i\xi}\right\},$$
$$\varphi(\alpha) = -\arg G_1(-\alpha + i0),\ a \leqslant \alpha \leqslant b,$$
$$G_1 = \frac{d^2+\xi^2}{h^2+\xi^2}\frac{\beta_{21}S(i\xi)}{AP(i\xi)} \to \frac{b^2(\xi^2+c_{R_1}^{-2})}{4(1-\nu)R_1\xi^4}\ \text{(homogeneous medium)} \tag{1.7}$$
$$G_1 = \frac{4(1-\nu_1)}{3-4\nu_1}\left(\frac{\xi^2}{b^2}+1\right)\left(1-\beta_{11}^{-1}\beta_{21}^{-1}\right)\ \text{(medium 2 rigid)}$$

The function $\varkappa_+(i\xi) \equiv \varkappa(i\xi) = \varkappa_-(-i\xi)$ is analytic in the plane ξ with the notch $[-ib, -ia]$. The carriers of the originals of the functions $\varkappa_\pm$ are concentrated in the regions $t/b \leqslant x \leqslant t/a$ and $-t/a \leqslant x \leqslant -t/b$, $\varkappa_\pm \to 1$ at $\xi \to \infty$.

The function $\varphi(\alpha)$ is determined by different expressions, depending on the relative location of the velocities c_{kj}, and it coincides with the analogous function for a homogeneous media [2, 3],

$$\varphi(\alpha) = \operatorname{arctg}\sqrt{\frac{\alpha^4}{(b^2-\alpha^2)(\alpha^2-c_{11}^{-2})}} \to \arcsin(\alpha/b),\ \nu_1 \to 1/2$$

for a rigid medium 2. In general, the function has discontinuous derivatives ($a < \alpha < b$). It accounts for refraction of longitudinal and transverse waves at the interface and generates numerous surfaces of discontinuity in the solution. These surfaces are wave fronts from the point source. However, being convoluted with a smooth load, they give a smooth solution. We will not analyze the wave pattern here (it is very complex). To facilitate the derivation of specific expressions for $\varphi(\alpha)$, we will note only that the quantities $\alpha\beta_{kj} > 0$, $i\beta_{kj}\beta_{mn} > 0$ on the right (left) side of the cuts in the upper (lower) half-planes $\xi = \xi' + i\alpha$; on the opposite sides of the cuts, they take values with the opposite signs.

The function $\varkappa(i\xi)$ may have root singularities. This is related to the fact that in the general case the ratio S/P does not contain a radical as the multiplier (or divisor), and we take an arbitrary radical in removing the branching in (1.6) (this radical does not generate singularities of the functions $\varkappa, \varkappa^{-1}$ only for a homogeneous medium). Using another radical, such as $\sqrt{a^2 + \xi^2}$, we obtain the following as a result of another function

$$\varkappa_0(\alpha) = \sqrt{\left|\frac{a-\alpha}{b-\alpha}\right|}\,\varkappa(\alpha) \quad (\operatorname{Im}\alpha = 0). \tag{1.8}$$

The arbitrariness in the method of eliminate branching does not affect the construction of the solution, but (1.8) and (1.9) will be considered in the analysis. Study of the function $\varphi(\alpha)$ and the Cauchy integral in (1.7) leads to the following results:

for a homogeneous plane

$$\varphi(\alpha) = 0, \ \varkappa(\alpha), \ \varkappa^{-1}(\alpha) \neq 0 \ (\alpha = a, \ b),$$

for rigid medium 2 in the sense $c_{21}^2\mu_1/(c_{22}^2\mu_2) \to 0$ or $c_{11} = c_{12}$

$$\varphi(\alpha) = \pi/2, \ \varkappa_0(\alpha), \ \varkappa_0^{-1}(\alpha) \neq 0 \ \ (\alpha = a, b), \tag{1.9}$$

while in the remaining variants $(\varkappa_1 = |b-\alpha|^{-1/2}\varkappa)$

$$\varphi(\alpha) = 0, \ \pi/2, \ \varkappa_1(\alpha), \ \varkappa_1^{-1}(\alpha) \neq 0 \quad (\alpha = a, b).$$

We will limit ourselves to the solution of the problem in the velocity range $0 \leqslant l^{\cdot} < c_P$ (cases 1 and 2) and $0 \leqslant l^{\cdot} < b^{-1}$ (case 3). Taking into account (1.5)-(1.7), we determine the functions $G_{\pm}$ with the equality

$$G_{\pm} = s^{-1/2}\frac{h \mp i\xi}{d \mp i\xi}\frac{\varkappa_{\pm}(i\xi)}{\sqrt{b \mp i\xi}}. \tag{1.10}$$

It follows from the formulas for the originals presented below that there is a solution to the factorization problem (1.4).

We reduce the original $g_+(x, t)$ to the following form by using formulas for inversion of transforms of the form $f_1(s)f_2(q/s)$ [3] and the rule for differentiation of convolutions of generalized functions [11]

$$g_+ = \frac{H(x)}{\pi\sqrt{\pi x}}\frac{\partial}{\partial t}\left[\int_a^{t/x} Q(v)H(b-v)\,dv + H\left(\frac{t}{x} - b\right)\int_b^{t/x} Q(v)\,dv\right]$$

$$= \frac{H(x)}{\sqrt{\pi x}}\frac{\partial}{\partial t}\left\{\left[1 - \frac{(d-h)\varkappa(d)H(dx-t)}{\sqrt{(d-b)(d-t/x)}}\right]H\left(\frac{t}{x} - b\right) + \frac{1}{\pi}H\left(b - \frac{t}{x}\right)\int_a^{t/x} Q(v)dv\right\},$$

$$Q(v) = Q\left(v; \frac{t}{x}\right) = \frac{h-v}{d-v}\frac{\varkappa(v)\sin\varphi(v)H(v-a)}{\sqrt{|(b-v)(t/x-v)|}} \quad \left(\varphi(v) = \frac{\pi}{2}, \ v > b\right). \tag{1.11}$$

All of the above operations are regarded as operations on generalized functions coinciding with the normal operations if the functions coincide with the normal functions; in particular, the values of the singular integral coincide with its principal value if the density is continuous in accordance with the Hölder condition [11]; by integration within the range from α_1 to α_2, we mean integration from $\alpha_1 - 0$ to $\alpha_2 + 0$.

Equation (1.11) is derived by means of the residue theorem. Since we will make repeated use of this method of isolating singularities of an integrand function and calculating integrals, we will examine its application in more detail. We will find the limiting value of the contour integral $C = C_R + C_V$ in the plane $Z = t/z$, $z = x + iy$ (Fig. 1) at $R \to \infty$ (C_R is a circle of large radius R with a fixed center) from the auxiliary function

$$Q_0(Z) = \frac{h-Z}{d-Z}\frac{\varkappa(Z)}{\sqrt{(Z-b)(Z-t/x)}} \sim \frac{1}{Z} \ \text{ for } \ Z \to \infty \ \text{(uniform)}.$$

We have

$$\varkappa(Z^{\pm}) = e^{\pm i\varphi(v)}\varkappa(v), \ a < v < b, \ \varkappa(Z^{\pm}) = \varkappa(v), \ v \notin [a, b],$$

$$\sqrt{(Z^{\pm} - b)(Z^{\pm} - t/x)} = \pm i\sqrt{(v-b)(t/x - v)},$$

$$b < v < t/x, \ Z^{\pm} = v \pm i0, \ Q_0(Z^+) - Q_0(Z^-) = 2iQ(v; t/x),$$

$$\int_C = \int_{C_R} + \int_{C_v} \xrightarrow[R\to\infty]{} 2\pi i + 2i\int_a^{t/x} Q(v)\,dv = 2\pi i \frac{(d-h)\,\varkappa(d)}{\sqrt{(d-b)(d-t/x)}} H\left(d-\frac{t}{x}\right).$$

The last equality was used in deriving (1.11). We similarly obtain the formula for the original

$$p_+ = \left(\frac{\partial^2}{\partial x \partial t} + \frac{1}{c_p}\frac{\partial^2}{\partial t^2}\right)\frac{H(x)}{\sqrt{\pi x}}\left\{\left[1-\sqrt{\frac{h-b}{h-t/x}}\frac{H(hx-t)}{\varkappa(h)}\right]H(t-bx) - \frac{1}{b}\int_a^{t/x}\frac{\sin\varphi(v)}{\varkappa(v)(h-v)}\sqrt{\frac{b-v}{t/x-v}}\,dv\,H\left(b-\frac{t}{x}\right)\right\}.$$

The formulas for the originals g_- and p_- are obtained from the expressions for g_+ and p_+ by replacing x by $-$ x. We develop the convolution:

$$\psi(x, t) = g_+ ** \tau_- = J_1 + J_2 + J_3,$$

$$J_1 = \int_{x-t/b}^{x} \frac{\tau_-(\xi, t-bx+b\xi)}{\sqrt{\pi(x-\xi)}}\,d\xi,$$

$$J_2 = -A_0\int_{x-t/b}^{x}\int_{\xi d}^{t_0}\frac{\tau_-'(\xi, \tau - xd + t)}{\sqrt{\tau-\xi d}}\,d\tau\,d\xi = -A_0\int_{t/d}^{t/b}(\xi d - t)^{-1/2}\int_0^{t_1}\tau_-'(x-\xi+\tau/d, \tau)\,d\tau\,d\xi,$$

$$A_0 = \frac{(d-h)\,\varkappa(d)}{\sqrt{\pi(d-b)}},\quad t_0 = (d-b)\,x + b\xi,\quad t_1 = \frac{t-b\xi}{1-b/d}, \tag{1.12}$$

$$J_3 = \int_a^h \frac{F_0(v)}{v\sqrt{\pi v}}\int_0^t \sqrt{\tau}\,\tau_-'\left(x-\frac{\tau}{v}, t-\tau\right)d\tau\,dv,$$

$$F_0 = \frac{1}{\pi}\int_a^v Q(y; v)\,dy\,H(b-v)\,H(v-a),\quad \tau_-'(f, g) = \frac{\partial\tau_-}{\partial g}(f, g)$$

(the symbol ** denotes convolution with respect to the variables t, x).

Since we realized factorization (1.4), the solution of the problem takes the form [2, 3]

$$\begin{gathered}\tau_+ = -p_+ ** [\psi H(x-l)],\quad f_- = Ag_- * * [\psi H(l-x)],\\ \sigma = \sigma^0 ** [\psi H(l-x)],\quad u = u^0 ** [\psi H(l-x)],\\ \sigma^0 = -A\frac{\partial w^0}{\partial x},\quad W^0 = \frac{DG_-}{S},\quad U^0 = \frac{iAEG_-}{S}.\end{gathered} \tag{1.13}$$

Before inverting the function W^0, we note that D ≡ 0, σ ≡ 0 for a homogeneous plane. In accordance with the inversion formulas in [3], we have

$$w^0 = \lim_{y\to+0}[W_1(z, t) - W_1(\bar{z}, t)],\quad W_1 = \frac{\partial}{\partial t}\frac{iD_0(Z)}{2\pi z}\frac{h+Z}{d+Z}\frac{\varkappa_-(Z)}{\sqrt{b+Z}} * \frac{t_+^{-1/2}}{\sqrt{\pi}},$$

where the symbol * denotes convolution with respect to the variable t; $t_+^{-1/2}$ is a generalized function [11]. The radicals β_{kj} enter into the ratio $D/S = D_0$ only in the form of the products $\beta_{kj}\beta_{mn}$, i.e.,

$$\begin{gathered}D_0 \to \Psi_0(v) \pm i\Psi(v),\quad Z \to Z^\pm = x \pm i0,\quad a < v = t/x < b,\\ D_0 \to \Psi_0(v) \mp i\Psi(v),\quad Z\to Z^\pm,\quad -b < v < -a.\end{gathered}$$

Here, $\Psi(v) = \Psi(-v)$ and $\Psi_0(v)$ are real functions which can be calculated from definitions (1.2) and (1.3), and the function $D_0(Z)$ is continuous in the transition across the real axis in the Z plane at the remaining points. The function G(Z) is broken up on the cut $-b \le v \le -a$ [due to $\varkappa(Z)$)] and on the cut $v < -b$ (due to $\sqrt{b+Z}$). As a result of the calculations we obtain

$$\sigma^0 = A\frac{\partial^2}{\partial x\partial t}\frac{1}{\sqrt{\pi x}}\left[\frac{1}{\pi}\int_0^{t/x}\frac{F_2(v)\,dv}{\sqrt{|t/x-v|}} + F_3\left(\frac{t}{x}\right)\right] \equiv A\frac{\partial^2}{\partial x\partial t}[w_1H(x) + w_2H(-x)],$$

$$F_2(v) = \frac{h+v}{d+v}\left[\frac{\psi(v)}{\sqrt{b+v}}H(v-a)\,H(b-v) + \frac{\Phi(v)}{\sqrt{b+v}}H(-a-v)\,H(b+v) + \frac{D(v)}{S(v)}\frac{H(-b-v)}{\sqrt{-b-v}}\right]\varkappa_-(v),$$

$$F_3(v) = \frac{2h\varkappa_-(h)\,D(h)\,H(v-h)}{\sqrt{b+h}(d+h)\,S'(h)\sqrt{v-h}},\quad \Phi = \Psi_0(v)\sin\varphi(v) - \Psi(v)\cos\varphi(v).$$

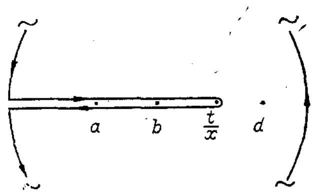

Fig. 1

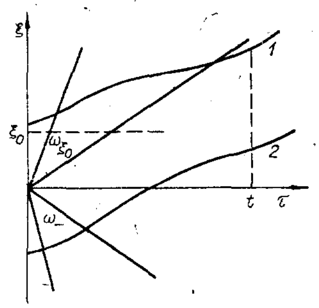

Fig. 2

The stress σ can be represented in the form of the sum

$$\sigma = \sigma^+ H(x-l) + \sigma^- H(l-x),$$
$$\sigma^+ = A\frac{\partial^2}{\partial x\,\partial t}\int_0^t\int_{\lambda_1}^{\tau/a} w_1(\tau,\xi)\,\psi(x-\xi,\,t-\tau)\,d\tau\,d\xi,$$
$$\sigma^- = \sigma^+ + A\frac{\partial^2}{\partial x\,\partial t}\int_0^t\int_{\lambda_2}^{0} w_2(\tau,\xi)\,\psi(x-\xi,\,t-\tau)\,d\xi\,d\tau, \quad (1.14)$$
$$\lambda_1 = [x - l(t-\tau)]H[x - l(t-\tau)],$$
$$\lambda_2 = -(\tau/a)H[\xi + l(t-\tau) - x] + [x - l(t-\tau)]H(\xi + \tau/a).$$

The contribution to the normal stress on the continuation of the cut is given by integration over the region lying above the curve 1: $\xi = x - \ell(t-\tau)$, $x > \ell(t)$ while the normal stress on the cut is given by integration over a polygonal region in the tetrahedron $\xi > 0$, $\tau > 0$ and over a curvilinear triangle in the tetrahedron $\xi < 0$, $\tau > 0$ bounded from below by curve 2: $\xi = x - \ell(t-\tau)$, $x < \ell(t)$ and the line $\xi = -\tau/a$ (Fig. 2).

2. Asymptotes of the Functions Near the Edge of the Crack. Let us study the behavior of the solution at $x \to \ell(t)$ at the moment of time t, when the quantities $\psi(x, t)$, $\ell(t)$ are functions of t and x which are continuous in accordance with the Hölder condition. For Hölder continuity of the function $\psi(x, t)$, it is sufficient to require that the function $\tau_-(x, t)$ have the same property throughout the domain.

The regions of integration in the first two formulas of (1.13) are triangles localized near the point $\xi = \tau = 0$ and disappearing at $x \to \infty$. Thus, the function $\psi(x-\xi, t-\tau)$ is brought out from under the integral sign [2], and we can integrate (and differentiate) the function remaining under the sign by means of the residue theorem. We present the final results:

$$\tau_+ \sim N_2(x-l)^{-1/2},\ x \to l+0;\ f_- \sim M(l-x)^{1/2},\ x \to l-0,$$
$$N_2 = \frac{(dl^\cdot - 1)\sqrt{1-bl^\cdot}}{\sqrt{\pi}\,(1-hl^\cdot)\,\varkappa(m)}\,\psi(l,t),\ M = -\frac{2S(m)\,N_2}{P(m)}, \quad (2.1)$$
$$l = l(t),\ l^\cdot = l^\cdot(t),\ m = 1/l^\cdot.$$

The integrals in the solution [2] are taken by the method of contour integration with the use of the residue theorem. The expressions for the constants A, B, $k(\ell^\cdot)$, and D are simplified:

$$A = \varkappa^{-1}(a),\ B = \varkappa(c),\ k = \varkappa^{-1}(1/l^\cdot),\ D = \varkappa(0). \quad (2.2)$$

We used the notation in [2] in Eqs. (2.2). With allowance for (2.2), the asymptotes $c_P \to c_R$, $c_S \to b^{-1}$ with the continuous convergence of the parameters of media 1 and 2, and the substitution $a \to b$ in the transition from a normal-rupture crack to a shear-rupture crack [3], the results in Parts 1 and 2 agree with the results in [2, 3].

It is more complicated to establish the asymptote for the function $\sigma(x, t)$ (the region of integration is not localized at $x \to \ell - 0$). It can be shown that $\sigma^+ = O(1)$ at $x \to \ell + 0$. We obtain the asymptote of the function σ^- on the basis of the representation (1.14). However, it is more easily obtained from

$$\sigma = -\frac{\partial}{\partial x}\,d_0 ** f_-; \quad (2.3)$$

$$d_0=\lim_{y\to+0}\left\{\frac{t}{2\pi z}[D_0(Z)-D_0(\bar{Z})]\right\}=-\frac{1}{\pi x}\frac{\partial}{\partial t}\left\{\psi\left(\frac{t}{x}\right)\left[H\left(b-\frac{t}{x}\right)H\left(\frac{t}{x}-a\right)-H\left(b+\frac{t}{x}\right)H\left(-a-\frac{t}{x}\right)\right]\right\}+D_\infty\delta(t)\delta(x),$$

$$D_\infty=\lim_{Z\to\infty}D_0(Z)\neq 0,\infty,\ \sigma^-=-\frac{\partial}{\partial x}\int\limits_{\omega_++\omega_-}d_0(\xi,\tau)f_-(x-\xi,t-\tau)\,d\xi\,d\tau,$$

$$\omega_+=\left\{\xi,\tau:\ \frac{\tau}{b}\leqslant\xi\leqslant\frac{\tau}{a},\ 0\leqslant\tau\leqslant t\right\},\ \omega_-=\left\{\xi,\tau:\ -\frac{\tau}{a}\leqslant\xi\leqslant-\frac{\tau}{b},\right.$$

$$\left.\xi\geqslant x-l(t-\tau),\ 0\leqslant\tau\leqslant t\right\} \tag{2.4}$$

[$\delta(t)$ is the Dirac delta function].

The measure of the region of integration ω_- approaches zero at $\varepsilon=|x-l|\to 0$. However, here $\operatorname{mes}\omega_+=O(1)$. We proceed as usual in the analysis of the asymptotic solutions of elliptic equations at branch points of boundary conditions and points of inflection. Specifically, we construct a section of the sector ω_+ with the straight line $\xi=\xi_0>0$, where ξ_0 is a sufficiently small (independent of ε) number, so that in region $\omega_{\xi_0}=\omega_+\cap\{\xi\leqslant\xi_0\}$ we can replace the function $f_-(x-\xi,t-\tau)$ by its asymptotic expression $f_0=M\sqrt{\varepsilon}$ (by virtue of continuity). The contribution to the general term of the asymptote is not given by integration over the region located above the straight line $\xi=\xi_0$, while the contribution to I_1 from integration over ω_{ξ_0}, as will be shown below, is independent of the selection of ξ_0

$$I_1=\frac{M}{2\pi}\frac{\partial}{\partial t}\int\limits_a^b\int\limits_0^{\xi_0}\frac{\Psi(v)\,dv\,d\xi}{\sqrt{\xi-x+l(t-v\xi)}}=-\frac{M}{4\pi}\int\limits_a^b\int\limits_0^{\xi_0}\frac{\Psi(v)\,l^{\cdot}(t-v\xi)\,dv\,d\xi}{[\xi-x+l(t-v\xi)]^{3/2}}.$$

For the functions $l^{\cdot}(t)$, continuous according to the Hölder condition, we have ($\gamma>0$)

$$l^{\cdot}(t-v\xi)=l^{\cdot}(t)+o(\xi^\gamma),\ l(t-\xi v)=l(t)-l^{\cdot}(t)v\xi+o(\xi^{\gamma+1}),\ \xi\to+0.$$

With the substitution $\xi=\varepsilon\eta$ and $\xi_0/\varepsilon\to\infty$, it is not hard to show that

$$\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\,I_1=-\frac{Ml^{\cdot}}{2\pi}\int\limits_a^b\frac{\Psi(v)\,dv}{1-l^{\cdot}v}.$$

We calculate the contribution to I_2 from integration over the region ω_- and we calculate the quantity I_3 — the result of convolution with the first term of (2.4):

$$I_2=\frac{Ml^{\cdot}}{2\pi\sqrt{\varepsilon}}\int\limits_{-b}^{-a}\frac{\Psi(v)\,dv}{1-l^{\cdot}v},\ I_3=-\frac{1}{2}\frac{M}{\sqrt{\varepsilon}}D_\infty.$$

The sum I_1+I_2 can be transformed by means of the residue theorem by choosing $Q_0=W_0(Z)\times(Z-m)^{-1}$ and the integration contour $C=C_R+C_V$, with the segments $[-b,-a[$ and $[a,b]$ on the real axis. As a result, we obtain

$$I_1+I_2=-\frac{M}{2\sqrt{\varepsilon}}(D_0(m)-D_\infty),$$

$$\sigma\sim\frac{N_1}{\sqrt{l-x}},\ x\to l-0,\ N_1=-\frac{D(m)}{P(m)}N_2=\frac{1}{2}\frac{D(m)}{S(m)}M. \tag{2.5}$$

We also determine the strain asymptote

$$\frac{\partial u}{\partial x}\sim E(m)N_2(x-l)^{-1/2}/P(m),\ x\to l+0;\ \frac{\partial u}{\partial x}=O(1),\ x\to l-0.$$

A change in the sign of the load is accompanied by a change in the sign of the coefficients N and M. At $N_1>0$, the assumption of continuity is violated (singular tensile stresses cannot be removed by finite static stresses) — a discontinuous region is inevitably formed. However, in the event of substantial external compressive forces, this region will evidently be the only such region, and it will be small and localized near the edge of the crack. Then the problem can be examined approximately by the method of combinable asymptotic expansions. Here, the internal problem reduces to the problem of the steady motion of a crack with a loaded discontinuous region at the tip, a contact section with slip about the edge of the crack, and an assigned asymptote at infinity (similar to the problem in [12]).

The results in [12] and a priori estimate of the capacity of the forces acting on the crack edges in the discontinuous region lead to the following important conclusion. The flow of energy to the edge of the crack from outside changes by an asymptotically small amount as a result of this capacity. The stress intensity factors obtained from the solutions of the external and internal problems will have the same absolute value but opposite signs — the discontinuous region functions as a switch which changes the signs of the stresses.

3. Special Cases of Loading. Crack Starting and Growth. The integrals (1.12) are taken (using contour integration) for dynamic loads of the simplest type. First let $\tau_- = H(t - \tau_0)H(\ell - x)$ — the problem of the diffraction of a stepped transverse wave on a movable notch (normal incidence). We find

$$\psi(x, t) = \psi^0 (t - \tau_0)_+^{1/2}, \quad N_2 = N^0 (t - \tau_0)_+^{1/2},$$
$$\psi^0 = \frac{2h\varkappa(0)}{d\sqrt{\pi b}}, \quad N^0 = \frac{2}{\pi}\frac{l^\cdot - c_P}{c_S - l^\cdot}\frac{\varkappa(0)}{\varkappa(m)}\sqrt{b^{-1} - l^\cdot}.$$

From here, it is easy to use superposition to obtain other results, such as for a linearly increasing load $\tau_- = (t-\tau_0)_+ H(l - x)$ $\psi = \frac{2}{3}\psi^0 (t - \tau_0)_+^{3/2}$, $N_2 = \frac{2}{3} N^0 (t - \tau_0)_+^{3/2}$ and for a triangular pulse $\tau_- = [a_1 t_+ - a_2(t - \tau_1)_+ + (a_2 - a_1)(t - \tau_2)_+]$ with a rise time $\tau_1 > 0$ and a total duration τ_2

$$N_2 = \frac{2}{3} N^0 \left[a_1 t_+^{3/2} - a_2 (t - \tau_1)_+^{3/2} + (a_2 - a_1)(t - \tau_2)_+^{3/2}\right],$$
$$0 < a_1 < a_2, \quad \tau_2 = a_2 \tau_1 (a_2 - a_1)^{-1}.$$

The maximum of the last expression occurs at the moment of time $t_m = \tau_1/[1 - (a_1/a_2)^2]$ ($\tau_1 < t_m < \tau_2$), and the problem of starting of the crack can be solved by comparing the value $N_m = N_2(t_m)$ with the limiting dynamic stress intensity factor during starting N_{2g}.

We will examine a load of the type $\tau_- = H(x - L + vt)H(l - x)$, moving at the velocity $v > 0$ over a notch from the edge. At $v > a^{-1}$, this models the incidence of a plane wave at an angle to the contact line with supersonic motion of the wake. We perform the calculations:

$$\psi = 2C\sqrt{\frac{x - L + vt}{\pi(1 + bv)}}\, H(x - L + vt),$$
$$N_2(v) = \frac{2}{\pi}\sqrt{\frac{(l - L + vt)(1 - al^\cdot)}{1 + bv}}\,\frac{dl^\cdot - 1}{1 - hl^\cdot}\,\frac{C}{\varkappa_0(m)} \to N_2(\infty), \quad v \to \infty,$$
$$C = 1 + \frac{(h/d - 1)\varkappa(d)}{\sqrt{1 + c_P/v}} + \frac{\sqrt{1 + bv}}{\pi}\int_a^b Q(y; 0)\arcsin\sqrt{\frac{1 - y/b}{1 + vy}}\,dy.$$

Analysis shows that $N_2(v) < N_2(\infty)$. We recall that it is necessary to put $h = b$, $h = d = b$ (cases 2, 3), $h = b$, $c_P = c_R$ (homogeneous plane) in all of the expressions.

Let us use the simplest mechanical criterion of fraction $N_2 \leq N_{2g} = \text{const}$ [we do not know of any data on the function $N_{2g}(\ell^\cdot)$]. Proceeding on the basis of the above relations for $N_2(\ell^\cdot)$, it is also possible to solve the equations for $\ell^\cdot$ [13] and to make the following qualitative conclusions (for nondecreasing loads). The crack remains stationary up to a certain moment of time and then accelerates ($\ell^{\cdot\cdot} > 0$). Meanwhile, $\ell^\cdot \to c_P$, $t \to \infty$ (cases 1, 2). To analyze motion in case 3, it is expedient to make the substitution $\varkappa(m)$ in accordance with (1.8), (1.9). We then find that the velocity $\ell^\cdot$ reaches the values b^{-1} after a finite time (the velocity of the shear wave is not critical for crack growth in case 3). On the other hand, $N_2 \to 0$, $N_1 = O(1)$ at $\ell^\cdot \to c_P$, $t = \text{const}$ (cases 1, 2), $N_2 = O(1)$, $N_1 = O(1)$ at $0 \leq \ell^\cdot \to b$, $t = \text{const}$ (case 3) for the above dynamic loads. This appreciably distinguishes the nonsteady motion from the steady motion.

We will analyze the establishment of a steady-state regime using the example $\tau_- = \delta(x - ct)H(t)$ ($0 \leq c \leq \ell^\cdot < c_P$). Then ($x' = x - ct$)

$$\psi = \frac{H(x')}{\sqrt{\pi x'(1 - bc)}}\left[\frac{1 - hc}{1 - dc}\varkappa\left(\frac{1}{c}\right) - H\left(b - \frac{t}{x}\right)\right] - \frac{A_0 H(dx - t)}{(1 - dc)\sqrt{dx - t}} - \frac{\partial}{\partial t}\sqrt{x'}\,H(x')\int_{t/x}^{b}\frac{F_0(y)\,H(y - a)\,dy}{\sqrt{\pi}(1 - cy)^{3/2}}. \quad (3.1)$$

At $t \to \infty$, $x <$ const + vt, $v < c_P$, and only the first term remains in (3.1). Assuming $\ell(t) \to L + ct$, $t \to \infty$, by inserting $x' = L$ into (3.1) we obtain $\psi(\ell, t)$ — the multiplier in the expressions for the coefficients N_2, M of (2.1) determining the steady-state value $N_2 = -(\pi\sqrt{L})^{-1}$.

For the law of motion $\ell = L + ct$, the asymptotes reach the steady-state régime after the shear and boundary waves from the movable point source pass the edge of the crack.

Thus, the formally obtained solution (1.13) satisfies the zero initial conditions and the restriction on the energy flow [4]

$$0 \leqslant W = -\frac{\pi \dot{l} \cdot N_2^2 S(m)}{2P(m)} < \infty$$

as well as the remaining conditions (1.1); σ, τ_+, f_-, and u are sufficiently smooth functions if the same is true of the input functions $\ell^\cdot$, τ_- (excluding the point $x = \ell$); the relationship between the coefficients N and M of (2.5), extracted directly from the solution, is analogous to the relationship for steady motions of the notch [4] (also see the analysis of the angular distribution of the functions in [4]); the solution in the limiting case $c_{k1} \to c_{k2}$, $\mu_1 \to \mu_2$ (k = 1, 2) coincides with the solution in [2].

The effect of the resistance of the notch edges to movement (which may vary broadly in its physical character) is actually accounted for in the form of a correction to solution (1.13), such as by introducing a shear stress equal to $\tau^0 = \tau^0(x, t; f_-, \sigma)$.

It should be noted that the study [14] obtained a criterion for the existence of a Rayleigh wave for contacting elastic bodies with slip.

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